Skew frequency curves in Biology and Statistics

by

J. C. KAPTEYN, Ph. D.; Sc. D.

Professor of Astronomy at Groningen.

1. Introduction. The substance of the following paper was delivered as a lecture before the scientific section of the Physical society of Groningen on 14 October 1916. It was meant as a popular exposition of the investigations contained in two more extensive papers published by the Astronomical Laboratory at Groningen, the first of which (by J. C. Kapteyn) appeared in 1903, the second (by J. C. Kapteyn and M. J. van Uven) in 1916 [1]. In what follows I will refer to these papers as *first* and *second paper*.

In writing my part of the second paper — and as a matter of fact even already in the first — I tried, as much as was in me, to avoid any superfluous mathematical development. I found however, that, if we wish to develop the theory in as rigourous a form as possible, it is not well possible to avoid some, to the general reader very formidable looking formulae. As I know very well that this fact will be much in the way of a somewhat extensive application of the theory, I resolved, even at the time of writing the second paper, to work out also a popular

[1]) Printed by Hoitsema Brothers, Groningen.

exposition of the theory in which all such mathematical development should be altogether banished.

The present paper is the outcome of this resolve. It covers the lecture given before the scientific section, but has been slightly extended and completed by the addition of the necessary tables; which will make the reader completely independent of the more extended technical papers. Meanwhile a popular account like the present does not, of course, pretend to supersede the more technical papers altogether. Leaving aside most of what seems to have more of a theoretical than of a practical importance (for the former I have still to refer the reader to our second paper) it tries to give a clear insight in the essential points of the method and to work these out with sufficient detail for practical application.

Before concluding the introduction I wish to repeat the words of the introduction to the 2nd paper:

"The main purpose of both papers — the finding something about causes — is no doubt an ambitious one. Indeed it may be well to warn expressly against too sanguine expectations. The theory necessarily starts from certain assumptions. These assumptions are probably not or not fully realized in nature. Therefore it is impossible to say *a priori* in how far our theory will apply to the cases offered by nature. The main ground for not being altogether sceptical lies in the fact that a close approach to the normal curve has already been found to occur frequently. Now our theory is only as it were an extension of the mathematical theory which leads to the normal curve and this extension starts from what is certainly in innumerable cases a "*vera causa*" *viz* that the "deviations" are dependent on the size already reached by the individuals. A reasoning like that of art. 9 of the first paper (art. 11 of the present one), shows this with perfect evidence.

Still the fact remains that the conclusions to which the theory leads must not be taken as well established facts but rather as *working hypotheses*".

Table I. Stature of 8585 men.

Height in inches.	Numb.	Frequ.	Scheme.	Height in fract. of average Stature.	Frequ.
57					
	2	0.000			
58					
	4	.000			
59			0.000		
	14	.002			
60			.002	0.86	
	41	.005			0.001
61			.007	.88	
	83	.010			.005
62			.017	.90	
	169	.020			.013
63			.037	.92	
	394	.046			.034
64			.083	.94	
	669	.078			.088
65			.161	.96	
	990	.115			.154
66			.276	.98	
	1223	.142			.198
67			.418	1.00	
	1329	.155			.200
68			.573	1.02	
	1230	.143			.166
69			.716	1.04	
	1063	.124			.085
70			.840	1.06	
	646	.075			.040
71			.915	1.08	
	392	.046			.011
72			.961	1.10	
	202	.023			.004
73			.984	1.12	
	79	.009			.001
74			.993	1.14	
	32	.004			
75			.997		
	16	.002			
76			.999		
	5	.001			
77			1.000		
	2	.000			
78					
Total . . average 67.5	8585	1.000			1.000

2. Normal curves. It has long been recognized that if we measure a great number of individuals and if then we plot the frequency with which the different statures occur, these frequencies arrange themselves in a regular curve. The following example will illustrate the fact. It summarizes the measures of the length of 8585 [1]) men between the ages of 23 and 50 years in Great Britain.

The meaning of this table is perhaps best seen by an example. From the first part of the table we learn that among a total 8585 men 169, that is the fraction 0.020 of the whole, have a height between 62 and 63 inches and further that the fraction 0.037 of the whole is below 63 inches. The name "Scheme" has been given by Galton to the curve which we get when we plot the numbers of the 4th col. as ordinates corresponding to the numbers of the first col. as abscissae. The curve is represented in fig. 1.

In what follows we will, following the general use, designate the abscissae by the letter x, the ordinates by the letter y.

In the second part of the table the statures have been expressed, not in inches, but in fractions of the average stature, which in the present case (see last line of table) turns out to be 67.5 inches.

In fig. 1 is also shown the frequency curve wich we get by plotting the frequencies of the 3d col. as y's over the numbers of the first Col. as x's.

It will be seen at once that whereas (to take an example) the frequency of stature below 65 inches is represented in the *frequency curve* by an area — the area of the curve below the ordinate of $x = 65$ — it is represented in the *scheme* by this ordinate itself. Quite generally frequencies are represented by areas in the frequency

1) Taken from: Report of the Brit-Assoc. 1883, p. 256.

curves, by lines in the scheme. This is what gives the scheme a great advantage in many cases.

The same regularity which we here find in the distribution of the frequencies of the different heights of men, we find back in numberless cases presented by nature. In order to see this regularity in its true light it is important to express the sizes of the measured property in fractions of the average amount, as was done in the second part of the preceding table.

Fig. 2 represents three of such cases all expressed in this way:

I. (Highest curve). Represents the same case as that of fig. 1 (stature of men).

II. Length of the lowest fruit on the main stem of 568 specimens of Oenothera Lamarckiana (H. de Vries, Ber. der Deutsch-Bot. Gesellsch., 1894, Bd. 12, Heft 7, p. 200).

III. (Lowest curve). Strength of pull of 519 males aged 23—26. (Galton, Natural Inheritance p. 199.)

The figure shows that in all these cases we get curves having the same characteristics.

a. They reach their maximum for the average value of x;

b. they are symmetrical with regard to this maximum;

c. from the maximum the curves very gradually and without intermission slope down to zero;

d. they meet the x axis tangentially.

In the meanwhile the several curves are still very distinct especially in the fact that the whole range of the deviations from the average value is very different.

All these characteristics remind us very strongly of the frequency curves of accidental observation errors. Gauss and others have derived the mathematical form of these "error curves". They show all the characteristics *a, b, c, d,* and, like our curves, are only different in range, the range of the errors being of course smallest in the case of the best observations.

A comparison of fig. 3 with fig. 2 well illustrates the anology of the two sorts of curves. Fig. 3 shows three "error curves" for observations of unequal precision [1]). It will be acknowledged that the similarity is very striking.

Indeed in a great many cases the differences of the frequency curves of nature with error curves are not greater than can be readily explained by the remaining uncertainties of the observations. Where, as in the case of our curve I, the number of measures is very considerable, a drawing on the scale of that of our figures almost fails to show any difference at all.

It is for this reason that the Gaussian error curve has come to the called the *normal* frequency curve.

3. Skew curves. In the meanwhile it is certain that we find in nature curves which diverge markedly from this normal form. As a striking instance may be considered the wealth-curve, that is the curve giving the frequency of the different amounts of property. I am not in the possession of the direct data for such a curve, but everybody realises that the most frequent amount of property is not very high. Let A be this most frequent amount. As the smallest amount is *zero*, the greatest deviation from the amount A on the one side is — A. On the other side it is of course immensily higher. The frequency curve therefore is small in extent on the lower side of the maximum, very extensive on the other. It thus must be a highly dissymmetrical curve.

As a rough substitute for the wealth curve we may perhaps use the frequency curve offered by the *valuation of house property (x). in England and Wales for the years 1885 and 1886*, as given by Pearson. Phil. Trans. Vol. 186, p. 396. Owing to want of detail in the data,

[1]) In order to insure the greatest similarity with fig. 2, the modules of precision were taken resp. 18.9; 5.96; 4.38.

there is a slight difficulty in drawing the curve quite empirically very near the value *zero*. For this neighbourhood I have taken the representation resulting from the discussion of this case in Example 3 (see farther on, art. 15). Fig. 4, dotted line, shows this curve.

Another good example is offered by Heymans' observations of the threshold of sensation. It is shown in fig. 5. 120 determinations were made of the minimum weight which still produces a sensation of pressure. The various determinations fall in the curve represented by the figure.

Here too it is evident *à priori*, that if p is the real value of the threshold, deviations on the negative side cannot exceed p, whereas on the other side there is no such limitation. Dissymmety of the curve seems therefore a priori probable.

4. Origin of normal curves. Coming back to the normal curves which, not-withstanding such cases as those just now considered, seem to dominate in nature, we are naturally led to the question; what is the reason of the widespread occurrence of just this curve?

In elucidation of this question I will quote in full the reasoning of the *1th paper* [1]).

Take the following example:

Suppose we have measured the diameters of a great number of ripe berries; that we have determined the frequency of diameters between 2.0 and 2.1; between 2.1 and 2.2 millimetres, and so on from the smallest of all the diameters to the largest one.

Suppose further that these frequencies arrange themselves practically in a *normal* curve.

---

[1]) The following pages (down to the end of art. 12) are practically litterally quoted from the 1st paper. My purpose in inserting so unusually long a quotation has been to make the present popular exposition complete in itself, so that the reader who wishes to acquaint himself with the method may find all he wants together.

How have we to picture to ourselves the way in which just this distribution and no other has originated?

Let us begin by imagining that on a certain date (say 1 May), shortly after the time of blossoming, we isolate a great number p of unripe berries of equal diameter. On that date there will fall a certain quantity of rain which causes some growth of the berries. The gain in diameter will not be the same, however, for all the berries. Some of the shrubs will be much exposed to the rain, others will be more or less screened. The difference of the soil will be cause that some plants will derive a greater benifit from the same quantity of water than others. Various causes in the plants themselves will favour some of the fruits more than others etc. etc.

Most probably, all the fruits will gain in diameter. For the present we will neglect the *mean* growth of the p berries, however, and confine our attention to the deviations from this mean growth.

Part of these deviations must necessarily be *positive* while another part must be *negative*. If we suppose, as the simplest, though certainly by no means probable case, thath half of the berries grow in diameter: mean growth $+ \Delta$, while for the other half this quantity is: mean growth $- \Delta$, then, owing to the rain on 1 May, and to this cause alone, we would have at the end of that date:

a. $\frac{1}{2}p$ berries whose diameters deviate $- \Delta$ from the mean,

b. $\frac{1}{2}p$ " " " " $+ \Delta$ " " " .

Now consider a second cause of deviation in operation at the same or at some other time.

For the sake of clearness let us take the sunshine of the 2nd of May. This sunshine will be profitable to the berries, but not for all in the same degree. We will suppose that its effect will be the same as that of the rain of the preceding day, that is to say, that it will cause half of the fruits to grow in diameter Δ more, half of the fruits to grow Δ less, than the average of the whole of the fruits.

Of the $\frac{1}{2}p$ berries (*a*) which at the the end of 1 May had diameters deviating — Δ from mean, half will now get a deviation of — 2Δ, the other half a deviation zero. Of the $\frac{1}{2}p$ berries (*b*) half will get the deviation zero, half the deviation $+ 2\Delta$, so that now we will have,

deviation	-2Δ	0	$+2\Delta$
number of berries	$\frac{1}{4}p$	$\frac{1}{2}p$	$\frac{1}{4}p$.

If we continue in this way and if we remark that the coefficients are no other than the binomial numbers obtained by the development of

$$\left(\tfrac{1}{2} + \tfrac{1}{2}\right)^2$$

we easely get to the conclusion that, in order to find the distribution of the diameters after the operation of n causes, we will only have to develop the binomial

$$\left(\tfrac{1}{2} + \tfrac{1}{2}\right)^n = \left(\tfrac{1}{2}\right)^n\left(1 + n + \frac{n(n-1)}{1.2} + \ldots . + n + 1\right) \ldots (1)$$

and we wil have

$$\left.\begin{array}{llll} \text{deviation} & -n\Delta & \text{in } \left(\tfrac{1}{2}\right)^n p & \text{cases} \\ \text{,,} & -(n-2)\Delta & \text{,, } \left(\tfrac{1}{2}\right)^n pn & \text{,,} \\ \text{,,} & -(n-4)\Delta & \text{,, } \left(\tfrac{1}{2}\right)^n p\frac{n(n-1)}{1.2} & \text{,,} \\ \ldots & \ldots & \ldots & \ldots \\ \text{,,} & +n(n-2)\Delta & \text{,, } \left(\tfrac{1}{2}\right)^n pn & \text{,,} \\ \text{,,} & +n\Delta & \text{,, } p & \text{,,} \end{array}\right\} \ldots (2)$$

Taking the deviations as abscissae and the frequencies as ordinates, we get a series of $n + 1$ points.

This figure is called a *Point Binomial.*

The continous curve which may be drawn through these points rapidly converges to the normal curve as n becomes larger and larger.

Prof. Galton (Natural Inheritance p. 63) has constructed an apparatus which gives an extremely instructive illustration of this way of generation of the normal curve.

5. Generalisation. In the meanwhile it must be evident, how utterly improbable the supposed effect of the various causes is. Not only will the several causes certainly *not* all have the same effect, but the influence of any one cause on different individuals will in general certainly *not* be to make half of them deviate a determined amount in one sense and the other half the same amount in the other sense. On the contrary, what we will look for is to find that the several individuals will derive the most various advantages of one and the same occasion, so that between the individual who makes the very best use of it and the individual who derives from it the smallest advantage, we will have individuals for whom the avantage has any of the infinite number of intermediate values.

Bessel has shown (Astr. Nachr. vol. 15, pp. 369—405) that, whatever be the effect of the various causes of deviation, as long as they are:

a. very numerous;

b. independent of each other;

c. such that the effect of any one cause is small as compared to the effect of all the causes together,

we will still obtain a curve which approximates the nearer to the normal curve the greater n is.

6. Dissymmetrical Point-Binomials. Bessel considers only causes, the effect of which is to give equal frequency to deviations of equal amount in the positive and negative direction (l.c. p. 378).

If now, with Quételet and Pearson we take into consideration causes which give a "tendency to deviation on one side of the mean unequal to the tendency to deviation on the other side" and if, as in the preceding case, we admit in the first place only causes, which, taken singly, produce no other deviations than those of $+\Delta$ or $-\Delta$; if further we assume that the frequency of the deviation $+\Delta$ stands to the frequency of the deviation

$-\Delta$ as $r:s$ (taking for the sake of convenience the numbers r and s so, that $r + s = 1$) then a reasoning like that of the preceding articles will lead to a binomial of the form $$p(r+s)^n \quad \ldots \ldots \quad (3)$$
and we will have, instead of (2):

$$\left.\begin{array}{llll} \text{deviation} & -n\Delta & \text{in } pr^n & \text{cases} \\ \text{,,} & -(n-2)\Delta & \text{,, } pnr^{n-1}s & \text{,,} \\ \text{,,} & -(n-4)\Delta & \text{,, } p\frac{n(n-1)}{1.2}r^{n-2}s^2 & \text{,,} \\ \ldots & \ldots & \ldots & \ldots \\ \text{,,} & +(n-2)\Delta & \text{,, } pnrs^{n-1} & \text{,,} \\ \text{,,} & +n\Delta & \text{,, } ps^n & \text{,,} \end{array}\right\} \ldots (4)$$

The corresponding point-binomial is again obtained by taking the deviations as abscissae and the number of cases as ordinates.

Now when r and s are very different, if we construct these point-binomials for very moderate values of n, we will find that they give a dissymmetrical arrangement for the deviations and this must be the cause why Quételet and Pearson have started from this form, to get an analytical representation of skew curves.

In order however to make the point-binominal approach a continuous curve and also in other respects to come nearer to the case of nature, we have to take n very considerable.

Now, as soon as we do this, we find that the point-binomial converges very rapidly, not to a *skew* curve, but to the *normal* one. The demonstration is not much more difficult than in the case of the symmetrical point-binomials and is virtually contained in Laplace: Theor. Analyt. des Prob. p. 301 etc. It is owing to this that Pearson does not obtain his curves by determining the limit of the point-binomial (4) for $n = \infty$, but by some indirect device.

If now, as before, we extend our consideration to causes the effect of which follows any arbitrary law, we will still find the *normal* curve as the limit of the frequency-curve [1]).

The simplest proof of this of which I know, is that given by Crofton Phil. Trans. vol. 160 p. 175. By such a proof the truth of Bessel's result is extended to causes, the effect of which is dissymmetrical.

The only conditions of its validity are the conditions *a*, *b*, *c* enunciated above.

7. Example of the way in which dissymmetrical Point-Binomials tend to become normal. It will be well to illustrate the way in which such a point-binómial as (4) tends to become a normal curve. For, if we consider that in such a binomial as for instance:

$$(\tfrac{1}{4} + \tfrac{3}{4})^n \quad . \; . \; . \; . \; . \; . \; . \; . \quad (5)$$

the first term is $(\frac{1}{4})^n$ and the last $(\frac{3}{4})^n$, so that, whereas a deviation of

$$- n\Delta \text{ will occur } (\tfrac{1}{4})^n \text{ times}$$

the deviation

$$+ n\Delta \text{ will occur } (\tfrac{3}{4})^n \text{ times,}$$

that is 3^n times as often, it seems at first sight difficult to imagine, how it is that the point-binomial corresponding to (5) still tends to become symmetrical. The seeming paradox is easily explained however.

In fig. 6 is given a representation of the point-binomials corresponding to (5) for the values $n = 4, 8, 12, 16, 20$.

In order that the continuous curves, which we draw through the points of the point-binomial be quite comparable, it is necessary to plot the coëfficients of (5) with intervals in the abscissae which are proportional to $\frac{1}{\sqrt{n}}$

[1]) In nearly every conceivable case the approximation will even be much *more* rapid than in the case of the point-binomial; see note of Bravais in Quetelet's Théor. des Prob. p. 421.

(it seems unnecessary to demonstrate this here) and I have accordingly taken them so.

The curves all begin at the same point of the axis of the abscissae marked 0 in the figure. The end-points lie at *a*, *b*, *c*, *d*, *e*. In reality, therefore, the curves are dissymmetrical; they always extend much further on the left side of the maximum than they do on the right-hand side; but the tail of the curve on the left-hand side is so close to the axis of the abscissae, that is to say, the frequencies of the smaller abscissae are so small, that even for moderate values of n they become quite insensible. For $n = 20$ I have drawn in the figure the normal curve having its maximum coincident with that of the point-binomial. As we see it is already all but wholly coincident with the dissymmetrical curve. For still larger values of n the dissymmetry would very soon disappear even in the most accurate drawings.

In fact, and here lies the explanation of the seeming paradox, for large values of n, the only part of the curves of any importance, is that on both sides of the maximum and this part becomes rapidly normal.

8. How do skew curves orginate? If therefore any causes whatever always produce normal curves, how do the skew curves originate? Though we may not at first sight see this, we may see at once the necessity of their existence.

Suppose, as before, that we find the diameters of certain ripe berries to be distributed in a normal curve.

Let us suppose further, what in most cases must be quite near the truth, that these berries are perfectly similar, and let the question be put: What will be the frequency-curve of the *volumes* of these berries?

It must be evident at once, that the form of this curve must be wholly determined by that of the diameters and a little reflection will easily prove that it cannot be a normal curve.

In the same way we would have found another skew curve, had we taken the surfaces of the berries as the object of our measurements, or if, after arranging the berries in the order of their magnitude, we had determined, for every one, the number of them contained in a fixed weight. We thus begin to realize that skew curves, far from being the exception, must be the rule in nature.

9. Reconciliation of this result with theory. The result here found seems in direct contradiction with the result of the theory, which, as we saw before, demonstrates that the effect of any causes whatever, satisfying the conditions *a, b, c,* will be the production of a *normal* curve.

For it seems evident at first sight that the causes which produce the variable growth of the diameters of certain berries, being identically the same as those determining the variable growth of the volumes, if the former satisfy the conditions *a, b, c,* the latter must do so of necessity.

The conclusion, evident as it might seem to be, is false.

If the effect of the sunshine on May 2nd may be considered to be absolutely independent of the effect of the rain on May 1st for *the diameters,* then the same effects are not independent for *the volumes.*

The reason lies in the fact that:

(A) { any causes whatever, the effect of which is different for different sized individuals, cannot be considered as independent.

We see at once the truth of this in our example. For the effect of the rain of May 1st has been to make the several berries *un*equal. Therefore, if the effect of the sunshine is different for individuals of different size, this effect on May 2nd will be another than it would have been had the rain of May 1st not existed and had, in consequence thereof, the berries not been unequal.

The effect of the various causes of growth can be independent, therefore, only in the case that the growth due to any one of these causes is the same for individuals of all sizes. — If we assume this to be the case for the *diameters*, then and then only, will the frequency-curve of these diameters become *normal*.

If however any causes have the same effect on the diameters of large and small individuals, then this same cause will have a *different* effect on the volumes of these same individuals.

Take for instance two berries of diameter 5^{mm} and two of diameter 10^{mm}.

Let the effect of a certain cause be to make them grow as follows:

				mm		mm		mm
1st	berry	from	diam.	5.00	to	5.01;	growth	0.01
2nd	„	„	„	5.00	„	5.02;	„	0.02
3d	„	„	„	10.00	„	10.01;	„	0.01
4th	„	„	„	10.00	„	10.02;	„	0.02

the effect on the diameters of the large and small berries being the same here.

Assuming the berries to be spherical, the growth in volume will be;

1st	berry	from	vol.	$\frac{1}{6}\pi\ (5.00)^3$	to	$\frac{1}{6}\pi\ (5.01)^3$;	growth	$0.75 \times \frac{1}{6}\pi mm^3$.
2nd	„	„	„	$\frac{1}{6}\pi\ (5.00)^3$	„	$\frac{1}{6}\pi\ (5.02)^3$;	„	$1.51 \times \frac{1}{6}\pi mm^3$.
3th	„	„	„	$\frac{1}{6}\pi\ (10.00)^3$	„	$\frac{1}{6}\pi\ (10.01)^3$;	„	$3.00 \times \frac{1}{6}\pi mm^3$.
4th	„	„	„	$\frac{1}{6}\pi\ (10.00)^3$	„	$\frac{1}{6}\pi\ (10.02)^3$;	„	$6.01 \times \frac{1}{6}\pi mm^3$.

the growth of the large berries in *volume* is thus found to be practically [1]) 4 times as large as that of the small ones.

After what has been said we thus find, that *if* the various causes of growth may be considered to be inde-

[1]) *Practically;* it would be rigorously so only for a growth in diameter infinitely small.

pendent in the case of the diameters, then they cannot longer be considered to be independent in the case of the volumes.

The difficulty of understanding, how it comes to pass that the volumes necessarily give a skew curve, in the case that the diameters give a normal one, is thereby removed.

10. Skew curves generated by causes whose effect depends on size. The net result of the preceding article may be considered to be that, wherever causes are at work, the effect of which depends on the size of the individuals, there we must expect skew curves. The consequence must be that, whereas the reasoning of arts 4—7 might seem to lead to the conclusion that the normal frequency-curves must be the rule in nature, we will conclude now that they must be the exception. For it will be clearly perceived that, even if we assume the effect of certain causes in producing deviations in certain quantities x, to be independent of the value of x, this cannot be the case with quantities proportional to x^2. x^3, $\frac{1}{x}$ etc., or more generally with any quantities whatever depending on x, which are not proportional to x itself.

We thus are led to consider the reverse of the former difficulty, that is: how is it, that normal curves, or at least curves but imperceptly different from normal curves are so common in nature.

The answer seems not difficult to give.

In the case of our example it would be as follows:

As long as the variations in the diameters of the berries are small as compared to the diameters themselves, the effect of the several causes of growth in volume, which depend on the size of the berries, must be little different too. Suppose for instance that the diameters of all our berries ranged only from 7 to 8*mm*, then the effect of the same cause on the volume of the smallest berry to that on the largest one will be as 49 to 64. This difference is still too small to cause any very marked skewness.

Now such a smallness of the variations as compared with the absolute size of the individuals, seems to be rather the rule in nature. The consequence will be, that, though in reality the curves will be skew, the difference from a normal curve will generally be very small.

The same reasoning explains, how we very generally find errors of observation distributed in normal curves. For in nearly all measurements the errors made will be incomparably smaller than the quantity measured.

There are some measurements however in which the errors become quite of the order of the quantity sought. Such for instance is the determination by observation of the *threshold of sensation*. Further on (see Example II, art. 15, tab. 5, fig. 5) I will give a series of measurements of this quantity, which shows that just in this case we find the errors of observation distributed according to quite another law than that of the normal curve. After what has been said, the fact has nothing very surprising.

11. Conclusion. Summing up, we find that causes independent of the size of the individuals produce normal curves, causes dependent on this size produce skew curves. The latter case must be the general one. There seems every reason to expect, however, that the skewness will be exceedingly small in many cases.

In several cases we feel at once that the effect of the causes of deviation cannot be independent of the dimension of the quantities observed. In such cases we may conclude at once that the frequency-curve will be a skew one. To take a simple example:

Suppose 10000 men to begin trading, each with the same capital; in order to see how their wealth will be distributed after the lapse of 10 years, consider first what will be their condition at some earlier epoch, say at the end of the fifth year.

We may admit that a certain trader A will then only possess a capital of 100 £, while another may possess 100.000 £.

Now if a certain cause of gain or loss comes to operate, what will happen?

For instance: let the price of an article in which both A and B have invested their capital rise or fall. Then it will be evident that, if the gain or loss of A be 10 £, that of B will not be 10 £, but 10.000 £; that is to say the effect of this cause will not be independent of the capital, but proportional to it.

I think that anybody will admit that it is somewhat the same with the effect of nearly any other cause. The effect will not be the same for the small and the large capitals. *Therefore:* the wealth curve will certainly *not* be a normal curve, but a skew one.

12. Skew-curve machine. On the same lines as Galton's apparatus for the normal curve, I have devised a machine, which will illustrate the genesis of the frequency-curve for the particular case that the effect of the various causes is strictly proportional to the absolute dimensions.

If we might admit that *all* the causes of gain and loss are proportional to the total capital, then this curve would give an exact representation of the wealth distribution. Probably it will give some rude approximation. The apparatus was constructed under the supervision and according to the directions of Prof. Moll and is embodied in the collection of instruments belonging to the botanical laboratory of our university. Those who are familiar with Galton's apparatus will readily understand the present one.

Like Galton's apparatus it is (see fig. 7) a frame glazed in front of about 5^{mm} depth. The pins however, which give equal deviations, have been replaced by what, for the sake of brevity, I will call *deviators*. They are pentagonal pieces of wood having one side horizontal, two sides perpendicular and the two upper ones inclined under a fixed angle (45°) to the horizon.

For the sake of compactness that part of some of the deviators at the right hand side of the rows, which is of no practical importance, has been cut off. The deviators of the same row have their bases on a horizontal line. The consecutive rows contain 3, 4, 5 . . . . of these deviators. Their breadth, measured from the middle of the channels between them, has been taken proportional to the distance of their tops from the left hand side of the frame, which is vertical. Likewise, though this is not essential, the breadth of channels between the consecutive deviators has been taken proportional to the distance of the middle of the channel to the same side of the frame. The tops of the deviators of any row have been placed just below the middle of the channels of the preceding row. With an infinite or at least a very great number of rows, we might, without changing the final result, have placed them in any way excentrically in regard to these channels.

At the top of the machine, just above the top of the middlemost deviator of the 1st row, has been placed a funnel. If we fill this funnel with fine sand, the grains of the sand will fall on this deviator and will be deviated one half to the right the other to the left. Arriving at the second row, both parts will again be devided in equal parts. The left hand part, however, will not now be so much deviated as the right hand part because of the different breadth of the deviators. As is evident from the construction of the apparatus the deviations will always be proportional to the distance from the left hand side of the machine. The deviators thus represent causes the effect of which is proportional to the distance from a fixed line; just as the causes of wealth may be said to be roughly proportional to the absolute dimension of this wealth.

After passing through several rows, therefore, the grains of sand must be distributed in a skew curve. This curve is rendered apparent as soon as all the sand is collected in the compartments which have been constructed below the last row.

In order to get workable dimensions for the deviators and the channels, the absolute dimensions have to be somewhat considerable. The total height of the outside of the frame is 104 centimeter and, even so, a few of the very narrowest channels had to be slightly widened to let the sand freely pass (this of course has no influence on the resulting curve).

Practical details. The deviators were made very exactly of equal thickness. They were glued to the glass which forms the front of the machine. The bottom is of tracing-cloth resting on a sheet of flannel, which is firmly pressed against it by a deal board. In this way the escape of the sand between the deviators and the bottom and front of the frame is pretty well avoided.

The beam on which the sand rests after having passed through the machine, can be taken off, in order to remove the sand for later experiments.

13. Questions raised by skew frequency curves. We have reached the result that normal causes are generated under the influence of causes which act with equal force on small and big individuals, whereas skew curves are generated under the action of causes which produce deviations depending on the size of the individual.

We are naturally led to the questions:

Given the observed skew frequency curve of the quantities x, *a*. is it possible to assign other quantities z, pure functions of x, which are normally spread?

b. is it possible to find the way in which the deviations depend on the size of the individual?

The importance, especially of the last question will be apparent, if we try to realize its true meaning in those cases where, as for plants and animals or parts thereof, there is *growth*.

For individuals of one determined size, under the influence of one cause, I call *growth* the average increase in size of all these individuals;

fluctuations the individual deviations from the average.

In most of the cases it will presumably be permissible to assume that the average of the fluctuations (these being all taken positively) is proportional to the growth; in other words that the average fluctuation is a certain percentage of the growth. In what follows we will assume that it is so. The consequence will be that the growth will be proportional to the total deviation and we may formulate our question *b*: is it possible to find the way in which the *growth* depends on the size?

Suppose the question solved we might then for instance find that the growth for plants of a certain size becomes all but zero and we would thus be led, by the simple consideration of the frequency curve of the plants harvested at an arbitrary epoch, to the conclusion that there is a period of rest in the growth at the time at which the plants reach such and such a size. Our attention might thus, in quite a new way, be drawn to interesting details in the process of growing.

Coming back now to our questions *a* and *b* themselves, I will develop their solution by the consideration of a particular example:

Suppose we have obtained from the observation of certain quantities x, the frequencies inserted in the second column of table 2.

Table 2.

x	Observed		z	z'	$\frac{0.1}{z'}$	$\frac{1}{10}x^3 - 1.50$	$\frac{0.667}{x^2}$
	Frequ.	Scheme					
0.0	0.017	0.017	− 1.500			− 1.500	
0.5	.001	.018	− 1.485	0.015	6.7	− 1.485	10.7
1.0	.006	.024	− 1.400	.085	1.18	− 1.400	1.19
1.5	.026	.050	− 1.162	.238	0.42	− 1.162	0.42
2.0	.111	.161	− 0.700	.462	0,22	− 0.700	0.22
2.5	.374	.535	+ 0.062	.762	0.131	+ 0.062	0.132
3.0	.420	.955	+ 1.200	1.138	0.088	+ 1.200	0.088
3.5	.045	1.000	∞				

It will be convenient to start, not from the frequency curve, but from the scheme. We therefore formed the scheme in the 3d Col. It has been represented in fig. 8.

Now the question *a* comes to this; can we find any quantities z, which are normally distributed and wich at the same time are pure functions of x, that is, are such that to any given value of x we can assign the corresponding value of z?

The solution of this question is extremely simple. And first: I maintain that the functions z must be such that,

a. they either continuously *in*crease,

b. or continuously *de*crease,

for increasing x's.

They cannot, for instance, begin by increasing and then afterwards change their increase for a decrease. For this would involve that, at the turning point, the z would not change at all for a certain change in x and, as will presently appear, (art. 14, Remark III) this must be considered as being impossible in nature.

Further; of the two cases *a* and *b* it will be sufficient to consider only the first. For if (case *b*) any quantities z, diminishing with increasing x, are normally distributed, then the quantities $-z$, which belong under case *a*, will also be normally distributed [1]).

Suppose, therefore, the z to increase regularly with the x and let x_1 and z_1 be two corresponding values and let it be remembered that as each individual x must have its corresponding individual z we must suppose the quantities x and z to be in equal number.

It follows that to any x below x_1 corresponds a value of z below z_1 and to any value of x in excess of x_1 a value of z in excess of z_1. For, if to any value x_0 below x_1 corresponded a value z_0 exceeding z_1, we would have, corresponding with the *increase* $x_1 - x_0$ of x, the *decrease* $z_1 - z_0$ of z, which is contrary to our supposition.

As therefore all the x below x_1, and no others, have their corresponding values of z below z_1, we conclude: that the total number of z below z_1 is equal to the total number of x below x_1.

Remembering the meaning of the *scheme*, we may express this by saying; if certain quantities z are pure functions of the quantities x, then *those values of x and z will correspond which in their frequency-schemes have equal ordinates.*

This being granted, let in fig. 8 on the left hand side

[1]) More generally: if any quantities z are normally distributed then it must be evident that 2 times, 3 times . . . . *b* times there quantities must be similarly distributed. Also that this distribution remains normal if we increase all our quantities by the same amount *a*. This then comes to saying that: if the z are normally distributed; the quantities $a + bz$ (where *a* and *b* may either be positive or negative) are also normally distributed. In reality therefore our problem must be considered to have an infinity of solutions. It is sufficient however to find one. From which we may, if we like, form all the others.

the scheme be drawn for *any* normally distributed quantities z. According to what has been said just now, *if* these quantities z are pure functions of x, those values of z and x must correspond, which have equal ordinates. Therefore, if AB is a line parallel to the x-axis, we must take the quantities

$$z = \text{OK} \text{ and } x = \text{OH}$$

as corresponding quantities.

This solves the first post (*a*) of our problem. For we see 1st that the z are normally spread quantities. In fact we *took* for them normally distributed quantities.

2nd they are pure functions of the x, for to each x we can assign its corresponding z. Our present example has been purposely so chosen that these quantities z are very simply algebraically expressible. It will be found (see last col. but one in table 2) that they are in reality equal to $\frac{1}{10}x^3 - 1.50$. In the present case therefore we are led to the conclusion that these quantities, therefore also the quantities x^3, are normally spread.

Suppose that the observed quantities x, (which are *not* normally spread) represent diameters of certain berries. We would thus be led to the conclusion that had not the diameters been measured, but the weights, we would very probably have found at least approximately a normal distribution.

[1]) We can now at once see the truth of what was maintained above that: we are exclusively led to functions z increasing continuously far increasing values of x. For as the frequency of any quantity from its lowest value up to any limit is higher, the higher this limit, the ordinates both of the observed and of the normal scheme necessarily increased with the abscissa. Therefore (see fig. 8) if OH that is x, grows, AH and consequently KB grows. But if BK grows z grows. We conclude that the x and the z grow at the same time. For the rest it is evident that if the z are normally spread, the $-z$ (which *de*crease with *in*creasing x) will also be normally spread.

Remark. We here found the value of z corresponding to any value of x from the figure. So for

$x = OH = +2.00$ we find $z = OH = -0.70$.

There is no difficulty in thus finding the values of the z corresponding to as many values of x as we please. But the same thing may be done even more conveniently and easily by a little table. Such a table will be found at the end of this paper (table 11).

With the argument z it gives the value of S (scheme). For the present purpose we have to enter this table with the given values of S (scheme) and to take out the corresponding values of the z. In fact the values z of table 2 have been thus computed.

14. Question b. It remains to find the way in which the deviations — therefore the growth and the average fluctuation — of the x depend on the size of the x.

The z being normally distributed, we know (see beginning of art. 13) that the deviations are independent of the size of the z. The growth of the bigger and of the smaller individuals are the same and the individual divergences from this mean growth are also the same.

From this it is easy to derive the growth of the x. This is perhaps most easily seen from the example summarised in table 2.

if x grows from 0.0 to 0.5 z grows from -1.500 to -1.485 that is

if at the average value 0.25 of x, x grows 0.500 z grows 0.015; similarly

if at the average value 0.75 of x, x grows 0.500 z grows 0.085;

if at the average value 1.25 of x, x grows 0.500 z grows 0.238 etc.

From this we find at once that

$$\text{if, near } x = 0.25, \ z \text{ grows } a, \ x \text{ grows } \frac{0.500\,a}{0.015};$$

if, near $x = 0.75$, z grows a, x grows $\frac{0.500\,a}{0.085}$;

if, near $x = 1.25$, z grows a, x grows $\frac{0.500\,a}{0.238}$ etc.

That is; in order to find the growth of the various x's we have in our table to make the column z' which gives the differences of the consecutive values of the z. If then the growth of any of the z under the influence of any one cause is a (the growth of all the z being equal), the growth of the x's will be

$$\frac{0.500\,a}{z'}$$

that is: the growth of any one x will be proportional to the quantity

$$\frac{1}{z'}$$

corresponding to that x.

For computing these reciprocals $\frac{1}{z'}$ a table has been given at the end of this paper (table 12). For reasons of convenience and because a constant factor is absolutely immaterial, the table gives $\frac{0.1}{z'}$ instead of $\frac{1}{z'}$.

In table 2 the values of the x corresponding to these values of $\frac{0.1}{z'}$ are those intermediate between the values of the first column and it is for this reason that these quantities (as also the z') have been printed between the lines.

I will call *reaction-curve* the graphical representation of the values of $\frac{0.1}{z'}$ or $\frac{0.1}{z'} \times$ const., because its ordinates represent the relative growth and fluctuations, in one word the relative amount of the deviations produced by any one cause for the individuals of different size x, that is the relative intensity with which individuals of different size *react* on the causes of growth.

The derivation of the *reaction-curve* constitutes the solution of part *b* of our problem. The solution of the problem proposed at the beginning of art. 13 is thereby completed. As already mentioned the present example was chosen in such a way that z becomes a simple *algebraic* function. The consequence is that $\frac{0.1}{z'}$ too becomes such a function. Indeed the reaction is inversely as the square of the dimensions as is shown by the last column which shows the values of $\frac{0.667}{x^2}$. These values are practically equal to those of the 6th col.; only the first is strongly divergent. As will be shown below (Remark II) this is entirely due to the unavoidable uncertainty of $\frac{1}{z'}$ near the limits of the frequency curve.

Remark I. In the preceding derivation is involved the tacit assumption that the rapidity of the growth at $x = 0.25$, 0.75 etc. is the same as the *average* rapidity between the limits $x = 0$ and $x = 0.5$; $x = 0.5$ and $x = 1.0$ etc. I think that in pretty well all cases of practice this assumption is completely allowable. If however in any particular case there should remain any doubt on the matter, recourse might be had to well known mathematical methods. For those not conversant with such methods I would recommend interpolation for smaller intervals of the x in the, thoroughly smoothed, scheme. So for instance in table 2 we would obtain by graphical interpolation by a large scale figure, the values of the ordinates of the scheme for $x = 0.25$, $x = 0.75$ etc. If thus operating with double the number of intervals, we are led to practically the same results as before (after multiplication of all the values by a constant, which in the present case will be 2) we may safely adopt the results. If there is a sensible difference we may repeat the process by the introduction of still smaller intervals.

Remark II. For those values of x for which the frequencies are very small — such as is usually the case towards the limits — the values of the z but particularly those of $\frac{1}{z'}$ are of necessity unreliable. If we want to know the degree of this unreliability we may change the given frequencies by amounts such as in our judgment may well subsist in our numbers, owing to incertainties of the measures, scarcity of data or other causes. So, for instance: if in table 2 we change the frequencies 0.001 and 0.006 between the values $x = 0.0$ and 0.5 resp. 0.5 and 1.0 to 0.000 and 0.007, the two first values of $\frac{0.1}{z'}$ *viz.* 6.7 and 1.18 will change to ∞ and 1.0.

An equal change in the considerable frequencies between $x = 2.0$ and 3.0 will hardly affect the values of $\frac{0.1}{z'}$ at all. Whereas the reaction curve, therefore, is very reliable for the middle values $x = 1.5$ to $x = 3.0$ it is enormously less so towards the extremities of the curve.

Remark III. Suppose that for a certain very small increase of x, the z did not change at all. The case would necessarily occur if the ordinates of the z curve were partly increasing with the x, partly decreasing. We would then have for the middle of the interval $z' = 0.000$ consequently $\frac{1}{z'} = \infty$. The reaction would thus be infinite. As such a thing cannot exist in nature, the supposed case cannot present itself.

15. Examples. From table 2 it will be seen how very easy the computation of both the normally spread function z and the reaction-curve $\frac{0.1}{z'}$ turns out to be. Having formed the observed scheme, we at once take out the values of z from table 11 (4th col.). Then, having formed the consecutive differences z' of these z (in the 5th col.) tab. 12 furnishes the values of $\frac{0.1}{z'}$ that is the ordinates of the reaction curve.

The following examples will serve to illustrate further both the process of computation and the conclusions to which this computation leads. The observed numbers and *full* numerical treatment will be given in the Appendix; the corresponding figures will be found at the end of the paper. In all the figures the frequency curve has been represented by a line in short dashes; the z by line in long dashes and the reaction curve by a continuous line. The figure for Example I shows a fourth curve, which is dotted, to represent the scheme. In order to get the figures on a suitable scale I have sometimes multiplied the numbers given in the Appendix for the frequency and the reaction curve $\left(\frac{0.1}{z'}\right)$ by some factor. For reasons that will appear further on such a procedure is not allowable for the z, at least if this curve must serve for the computation of the *quartiles* (see art. 17).

In the treatment of the observations I have sometimes thoroughly smoothed the frequency curve before using it for further work (see the computation in the Appendix). Of course the computer shall take good care not to smooth any trait out that he thinks really indicated by the observations. It is only the quite accidental irregularities that ought to be got rid of in this way. These irregularities are simply the consequence of an insufficiency in the number of observations. Where this number is very considerable all smoothing is rather to be avoided. The same holds for cases (as in our 3d Example below) where, by some inadequacy in the observed numbers, it is somewhat uncertain, *a priori*, *how* we ought really to draw the frequency curve.

A moment ago (prec. art. Remark II) we drew attention to the relatively great uncertainties, consequently irregularities, we have to expect in the reaction curve, especially towards the limits of the curve. These irregularities would come out smaller if the intervals in the x were taken greater. Such extension of the intervals being generally objectionable on the grounds mentioned above in remark I, the best way to act seems to be to smooth the reaction curve, graphically or otherwise. In the figures accompanying the following examples I have drawn such smoothed curves, but have left visible the points obtained directly from the computation. In fact, I simply drew a somewhat smooth curve passing as nearly as possible through the whole of these points, taking into account of course the very great uncertainty of the extreme points.

Example I. *Stature of 8585 men* (tab. 4 Fig. 1).

The fig. shows that the reaction curve is a *straight line* parallel to the x-axis. We conclude at once that the distribution is a *normal* one (for it means that the reaction, that is the deviations, are independent of the size x).

As must be always the case in the circumstances, the curve of the z is also a *straight line*, which however is inclined.

Example II. *Threshold of sensation* (tab. 5, fig. 5) taken from 1th paper p. 25. The observations are those of Prof. G. Heymans of the minimum weight which still produces a sensation of pressure. From the figure we see that the reaction curve is an inclined straight line, passing through the origin. The reaction is thus found to be proportional to the dimension x, that is in the present case: if under the influence of certain causes the threshold is high, a furher cause will have a greater effect than in the case that the momentary threshold were low, the effect being proportional to the momentary threshold itself.

We thus are led, by the simple consideration of the frequency curve, to the law of Fechner—Weber (see also 1st paper p. 42).

In all cases like the present in which the reaction curve is an inclined straight line, the curve of the z is a logarithmic one. Mathematically this is proved with the utmost ease. As we wish to avoid mathematical considerations however, the result may be here taken on trust [1]).

If therefore, we had treated as observed quantities, *not* the minimum weights which still produce a sensation of pressure, but the logarithms of these quantities, we would have been led to a *normal* curve.

Example III. Valuation of house property in England and Wales, years 1885—1886 as given by Pearson. Phil. Trans. Vol. 186 p. 396 (tab. 6, fig. 4).

This is the curve which in art. 3 was considered as a rough substitute for the wealth-curve. The fig. shows that the reaction curve is again well represented by an inclined straight line which passes somewhere near the origin. We conclude that the reaction is approximately proportional to the degree of wealth reached, just what, according to art. 11 we had to expect.

The curve of the z must again be a logarithmic one [1]).

This is all that can be maintained. Whether the reaction curve passes absolutely through the origin or not cannot be decided with any certainty. This is not owing to any defect in the method but to a defect in the data. The method tries to solve the problem: Given the observed frequency curve, find the reaction curve. In the present

---

[1]) If the reaction curve passes through the origin the equation of the z curve will be $z = \log. x$; if it cuts the x axis at $x = k$, the equation will be $z = \log. (x - k)$.

case the observation does *not* furnish the complete frequency curve. We have *not* the number of houses of value between 0 £ and 1 £; between 1 £ and 2 £ etc. We know only that below 10 £ the total number is 3 175 000, which is more than half of all the houses together. The consequence is that really the reaction curve can be assigned *with certainty* only from $x = 15$ £ unwards.

As, from this point on, this curve is evidently well represented by a straight line, we are, however, naturally led to assume that it will still be represented by this line for lower values of x. That therefore also the z curve will be logarithmic throughout. In this supposition we find (and for this interpolation it seemes somewhat better to rely on the simple mathematical computation given 1st paper p. 42) that the reaction curve cuts the x-axis at $x =$ £ 2.2.

Even this result, uncertain as it still is, is in accordance with what we should expect. For it is evident that there is a lower limit different from zero to the value of a house. Our result places this limit at 2.2 £ which seems reasonable enough, though of course we lay no stress whatever on the accuracy of this determination. As soon as more detailed data for the very low values shall be available, the last remaining uncertainty will be removed.

The frequency curve as shown in our figure has been drawn for the very small values of x, in accordance with the above supposition, that is to say in the supposition that for these small values too the log. $(x - 2.2)$ are normally distributed.

Example IV. Diameter of Spores of Mucor Mucedo, measured by Mr. G. Postma in the botanical laboratory at Groningen (unpublished) (tab. 7, fig. 9).

What draws the attention in the frequency curve is the enormous accumulation of individuals near $x = 20.5$. Corresponding therewith we find a very strong minimum in the reaction curve. We thus get an indication that at about the time when the diameter of a spore becomes about 18 or 19 units, there occurs a period of relative rest in their growth. I find just the same thing in the spores of Mucor Mucelagineus which have also been measured by Mr. Postma. As has already been pointed out (art. 1), we have to consider such a result rather as an indication (as a working hypothesis) than as a well established fact. In the present case there is outside evidence for the belief that our interpretation is the correct one. This evidence, to which my attention was kindly drawn by Prof. Hugo de Vries, is to be found in the result, arrived at already in 1884 by Prof. Errera, that there is a period of rest in the growth of the *sporangia* of some of the fungi of the same family.

Much weight ought not to be attributed to the downward slope of the reaction curve at both its extremeties. As already explained (art. 14, Remark II) the uncertainties near the limits are usually very considerable. They are so in the present case.

The fact here found, that to an abnormal accumulation somewhere in the frequency-curve there corresponds a minimum in the reaction curve, is general. The converse holds too; wherever there is in the frequency curve an abnormal depression, there we will find high ordinates in the reaction curve.

A good illustration is furnished by

Example V. Length of wheat-ears, grown under unfavorable circumstances (closely sown in poor soil), measured by Dr. C. de Bruyker (Handelingen 13e Vlaamsch Nat, en Geneesk. Congr. p. 172) (tab. 9, fig. 10).

The frequency curve is double topped. It is usual in such cases to conclude that we have to do with hybrids or with a mixture of two different species. In the present case there seems to be no reason whatever for such a supposition. Turning to our solution we find that the reaction curve shows a growth, which, for the smaller individuals is exceedingly small. At about size $x = 35$, this growth begins to increase with great rapidity. It rises to a high maximum for sizes between 50 and 70 mm, after which it again diminishes.

The whole case shows the greatest analogy with the next example and the explanation suggested by the reaction curve is much the same.

Example VI. [1]) (Stalk-length of *Linum crepitans*, measured at a moment in which the growth had not yet ceased by Miss A. Haga (Tab. 8 Fig. 11).

The frequency curve is again two topped. It might be described as a fairly common sort of curve with an enormous accumulation near the lower extremity.

About the treatment of this curve I will quote the words of 2nd paper p. 68. „As this might be a good test „case, we requested that no particulars should be com- „municated before we had derived the normal function (z) „and the reaction curve $\left(\frac{0.1}{z'}\right)$ in the ordinary way."

As a consequence we knew nothing of the nature of the object measured, safe that (as the numbers came from the botanical laboratory) they were in all probability relative to plants or parts thereof.

The reaction curve found and shown in the figure „starts from zero and then rises extremely abruptly. A „maximum however is soon reached at about $x = 27$, „after which it steadily decreases, so that the reaction „(growth) for $x = 100$ is already below half what it is at „maximum.

„The meaning of this is of course, that the individuals „evidently have great difficulty in starting their growth. „There seems to be an almost insuperable impediment „against beginning growth. Those individuals however,

[1]) Kindly communicated by Miss Dr. Tammes.

„who succeed in overcoming the first difficulty then begin „to grow very rapidly indeed, the rapidity increasing till „the size 27 is reached. After that the growth begins to „diminish; it gradually decreases, to only half of the „maximum growth for the individual of size 100 and to „one eighth of the maximum growth for individuals of „size 170.

„All this proves to be in good agreement with what „has been really observed. Dr. Tammes writes: „„the „„case I sent you is as follows: the quantities communi- „„cated are stalk-lengths of *Linum crepitans*, a variety „„of the ordinary flax. They were measured, at a moment „„in which the growth had not yet ceased, by Miss „„A. Haga. The seeds were sown in a great deep „„flower-pot. Their number was purposely taken very „„high, so that they were *extremely crowded*. At starting, „„therefore, the difficulty for each seed was to get a „„root into the soil. It seems allowable to assume that „„all seeds germinated. This has necessarily entailed an „„intense struggle and many individuals must not have „„succeeded or not sufficiently succeeded. For those who „„really got their root in the soil there now came a good „„time. There was plenty of food for a good many of „„very small plants. The case however changed when the „„plants, becoming greater, required more room. Then a „„second struggle ensued, *viz* the struggle for the available „„food in the too narrow room. The plants now became „„more and more impeded in their growth.

„„It seems to me that the conclusions from your curve „„are well in accordance with the facts.''''

16. Proportional curves.

What becomes of the frequency curve:

a. if for any one cause the reaction becomes λ fold;

b. if — the average reaction or deviation remaining equal — the *number* of causes grows in the proportion of $1 : \lambda$?

In the 2nd paper (p. 27) these curves were called *proportional* curves resp. of the *first* and of the *second* kind.

A distinction between the two kinds simply by the aid of the given frequency curves is impossible. Whether it will ever be possible to obtain the data necessary for such a distinction I do not know. For the present exposition it may at all events be sufficient to treat only the *first* kind of curve, referring those who might be interested in the curves of the second kind to the 2nd paper (p. 27).

As for any one cause the reaction becomes λ fold, the total reaction, that is the ordinate $\frac{1}{z'}$ if the reaction-curve, becomes λ fold. Therefore: *The ordinates of the reaction curves of proportional frequency curves are proportional* [1].

If the quantities corresponding to the λ fold causes are distinguished by the suffix λ, we get for the numerical expression of this property:

$$(a) \qquad \frac{1}{z_\lambda'} : \frac{1}{z'} = \lambda \text{ or } z'_\lambda = \frac{1}{\lambda}.z'.$$

I found this criterium of proportionality satisfied, with surprising approximation, in the case of the summer and winter barometerheights at den Helder, the data for which I owe to the courtesy of my friend Dr. v. d. Stok.

The observed frequencies, as well as the values of the quantities z and z' computed from them, will be found in tab. 10.

The last column shows the proportions

$$\frac{z'_w}{z'_s}$$

It is true that these still show small irregularities, but they are not greater than might have been expected

[1] This holds for proportional curves of both the 1st and the 2nd kind. In the first the proportion is as $1 : \lambda$ in the second as $1 : \sqrt{\lambda}$ (see 2nd paper).

(compare what has been said in Remark II art. 14) and do not show a well marked systematic change. In fact we may say with considerable approach to truth that the proportion is equal to the average value

$$0.544$$

throughout.

The conclusion to which we are thus led would be, that the difference in the distribution of the summer and winter barometerheights can be explained by assuming that they are governed by the same causes, which, however, in summertime act with an intensity of only about $54\frac{1}{2}$ percent the intensity in winter time.

It may contribute to a better understanding of the meaning of proportional curves, if we compute the frequency curve of the summer barometer readings theoretically from the winterreading. This computation offers no difficulty provided we first derive empirically *two* numbers from a comparison of the summer and winter observations. The first is the number $\lambda = 0.544$ already fouud. This is sufficient for the computation of the z'_s by

$$(b) \qquad z'_s = \frac{z'_w}{0.544}.$$

The further computation now becomes the inverse of that followed before, when we derived the z' from the observations. From the z' (see tab. 9) we first obtain the z. From these we then derive the values of the scheme and these finally yield the frequencies.

In passing from the z'_s to the z_s we will want the *second* of the necessary numbers. For as the z' are simply the differences of the z, in order that the z' may be $\frac{1}{0.544}$ times greater, the z themselves must be $\frac{1}{0.544}$ times greater. Besides, however, the z may all be increased by the same amount A. For it is evident that such an increase will have no influence on the differences z'. We thus have

(c) $$z_s = A + \frac{z_w}{0.544}.$$

This constant A then is the second quantity which we

Table 3.

Bar. heights in mm.	Frequency Obs.	Frequency Comp.	0 — C.
736.5			
738.5	0.0000	0.0001	— 0.0001
740.5	.0004	.0003	+ .0001
742.5	.0011	.0008	+ .0003
744.5	.0024	.0021	+ .0003
746.5	.0061	.0048	+ .0013
748.5	.0106	.0103	+ .0003
750.5	.0194	.0205	— .0009
752.5	.0353	.0377	— .0024
754.5	.0583	.0590	— .0007
756.5	.0850	.0835	+ .0015
758.5	.1103	.1110	— .0007
760.5	.1321	.1322	— .0001
762.5	.1401	.1398	+ .0003
764.5	.1369	.1315	+ .0054
766.5	.1101	.1128	— .0027
768.5	.0774	.0786	— .0012
770.5	.0446	.0466	— .0020
772.5	.0207	.0203	+ .0004
774.5	.0074	.0063	+ .0011
776.5	.0015	.0015	.0000
778.5	.0003	.0003	.0000
780.5	.0000	.0000	.0000
Totals	1.0000	1.0000	

have to borrow from the observations. I find that every thing is best represented if we take

(*d*) $A = 0.016.$

With this value formula (c) furnishses all the z_s.

These being obtained we find the numbers in the column "Scheme" by such a table as tab 11. In the present case it is necessary to use 4 decimals and I therefore made use of the table at the end of the 2nd paper. — Having got the scheme, we get the frequency curve by taking the differences between consecutive values.

The results obtained in this way are as shown by table 3, third column. The second shows the observed frequencies. The agreememt is surprising.

Remark 1. If by such agreement we feel convinced that we have really to do with proportional curves and if — by some independent means — we could be sure that the proportionality were of the first kind, then the curious result would follow, that we could determine the undisturbed barometer-height at den Helder, that is the reading the barometer would show in the absence of any perturbing factors. We must however refer to the 2nd paper for this matter.

Remark 2. All normal curves may be considered as proportional curves.

17. Medians and quartiles.

If we call

$x_{0.25}$ the x for which corresp. value in scheme is 0.25
μ „ „ „ „ „ „ „ „ „ 0.50
$x_{0.75}$ „ „ „ „ „ „ „ „ „ 0.75

then, according to difinition of the scheme

one fourth of all the x lie below $x_{0.25}$
„ „ „ „ „ „ „ between $x_{0.25}$ and μ
„ „ „ „ „ „ „ „ μ „ $x_{0.75}$

and according to usage

(e) $\begin{cases} \mu \text{ is called the } \textit{median}; \\ \mu - x_{0.25} \text{ is called the } \textit{1st quartile}; \\ x_{0.75} - \mu \text{ „ „ „ } \textit{2nd} \text{ „} \end{cases}$

A slight interpolation in our tables will thus yield these quantities.

If we wish to take them out of our figures we have to consider that

to value 0.25 in scheme corresp. the value $z = -0.4769..$
" " 0.50 " " " " " $z = 0.0000..$
" " 0.75 " " " " " $z = +0.4769..$

Therefore the values $x_{0.25}$, μ, $x_{0.75}$ will be the x for which z is resp. $-0.4769...$, $0.0000...$ and $+0.4769..$ In the figures the horizontal lines representing these values have been drawn slightly heavier than the rest. For the points of intersection of the z line with these heavier lines we have thus to read off the abscissae. They are at once the values of $x_{0.25}$, μ and $x_{0.75}$ and we have the median and quartiles by (e).

Table 4. Example I. Stature of 8585 men.
(fig. 1.)

x in inches.	Frequ. see tab. 1.	Scheme.	z by tab. 11.	z'.	$\frac{0.1}{z'}$ by tab. 12.
59					
	0.002				
60		0.002	— 2.03		
	.005			0.29	0.34
61		.007	— 1.74		
	.010			.24	.42
62		.017	— 1.50		
	.020			.24	.42
63		.037	— 1,26		
	.046			.28	.36
64		.083	— 0.980		
	.078			.280	.36
65		.161	— 0.700		
	.115			.280	.36
66		.276	— 0.420		
	.142			.273	.37
67		.418	— 0.147		
	.155			.277	.36
68		.573	+ 0.130		
	.143			.274	.36
69		.716	0.404		
	.124			.299	.33
70		.840	0.703		
	.075			.267	.37
71		.915	0.970		
	.046			.28	.36
72		.961	1.25		
	.023			.27	.37
73		.984	1.52		
	.009			.22	.45
74		.993	1.74		
	.004			.20	.50
75		.997	1.94		
	.002			.25	.40
76		.999	2.19		
	.001				
77		1.000			
	.000				

Table 5. Example II. Threshold of Sensation.
(fig. 5.)

x in decigr.	Numb.	Frequ,	Smoothed.	Scheme.	z by tab. 11.	z'	$\frac{0.1}{z'}$ by tab. 12.
0.5	1	0.008	0.008	0.008	— 1.70	0.58	0.172
1.5	6	.050	.049	.057	— 1.12	.572	.175
2.5	23	.192	.162	.219	— 0.548	.405	.247
3.5	21	.175	.201	.420	— 0.143	.325	.308
4.5	21	.175	.181	.601	+ 0,182	.280	.357
5.5	15	.125	.142	.743	0.462	.245	.408
6.5	15	.125	.098	.841	0.707	.200	.500
7.5	3	.025	.059	.900	0.907	.188	.532
8.5	7	.058	.039	.939	1.095	.175	.571
9.5	3	.025	.025	,964	1.27	.16	.625
10.5	2	.017	.014	.978	1.43	.14	.715
11.5	0	.000	.009	.987	1.57	.13	.77
12.5	1	.008	.005	.992	1.70	.12	.83
13.5	1	.008	.003	.995	1.82	.13	.77
14.5	0	.000	.002	.997	1.95	.08	1.25
15.5	0	.000	.001	.998	2,03	.16	0.62
16.5	1	.008	.001	.999	2.19		
17.5	0	.000	.001	1.000	∞		
Total	120						

Table 6. Example III.
Valuation of house property.
(fig. 4.)

x, unit 1 £.	Numb. in thousands.	Frequ.	Scheme.	z by tab. 11.	z'.	$\frac{0.1}{z'}$ by tab. 12.
0			0.000	$-\infty$		
	3175	0.544			∞	0.00
10			.544	+0.078		
	1451	.249			0.500	0.20
20			.793	.578		
	441.6	.076			.217	0.46
30			.869	.795		
	259.8	$.044^{5}$			.168	0.59^{5}
40			$.913^{5}$	.963		
	151.0	.026			.134	0.75
50			$.939^{5}$	1.097		
	90.4	$.015^{5}$			.10	1.00
60			.955	1.20		
	62.4	.011			.09	1.1
70			.966	1.29		
	41.7	.007			.07	1.4
80			.973	1.36		
	27.4	$.004^{5}$			.06	1.7
90			$.977^{5}$	1.42		
	19.9	$.003^{5}$			.05	2.0
100			.981	1.47		
	109.7	.019				
			1.000	∞		
Total	4829.7					

Table 7. Example IV.
Diam. of Spores of Mucor Mucedo.
(fig. 9.)

x unit 3.27 μ	Numb.	Frequ.	Smooth.	Scheme.	z by tab. 11.	z'.	$\frac{0.1}{z'}$ by tab. 12.
10							
	3	0.009	0.004				
11				0.004	— 1.88		
	3	.009	.006			0.23	0.44
12				.010	— 1.65		
	2	.006	.011			.21	.48
13				.021	— 1.44		
	7	.021	.020			.21	.48
14				.041	— 1.23		
	11	.033	.028			.18	.56
15				.069	— 1.05		
	12	.036	.039			.175	.57
16				.108	— 0.875		
	25	.076	.058			.188	.53
17				.166	— 0.687		
	26	.079	.077			.195	.51
18				.243	— 0.492		
	26	.079	.108			.183	.55
19				.351	— 0.309		
	50	.152	.151			.312	.32
20				.502	+ 0.003		
	106	.321	.314			.630	.16
21				.816	0.633		
	33	.100	.098			.332	.30
22				.914	0.965		
	10	.030	.037			.205	.49
23				.951	1.17		
	6	.018	.023			.20	.50
24				.974	1.37		
	7	.021	.015			.25	.40
25				.989	1.62		
	3	.009	.007			.32	.31
26				.997	1.94		
			.003				
				1.000			
Total	330						

Table 8. Example VI.
Stalk-lengths of Linum crepitans.
(fig. 11.)

x in mm.	Numb.	Frequ.	Smoothed.	Scheme.	z by tab. 11.	z'.	$\frac{0.1}{z'}$ by tab. 12.
0				0.000	$-\infty$		
	148	0.111	0.111			∞	0.0
5				.111	— 0.863		
	15	.011	.011			0.040	2.5
10				.122	— .823		
	9	.007	.008			.026	3.85
15				.130	— .797		
	8	.006	.007			.022	4.55
20				.137	— .775		
	7	.005	.006			.019	5.3
25				.143	— .756		
	10	.007	.006			.018	5.6
30				.149	— .738		
	9	.007	.006			.020	5.0
35				.155	— .718		
	9	.007	.007			.020	5.0
40				.162	— .698		
	10	.007	.008			.023	4.35
45				.170	— .675		
	14	.011	.009			.025	4.0
50				.179	— .650		
	13	.010	.010			.028	3.6
55				.189	— .622		
	13	.010	.011			.027	3.7
60				.200	— .595		
	21	.016	.013			.033	3.0
65				.213	— .562		
	16	.012	.014			.032	3,1
70				.227	— .530		
	22	.016	.015			.035	2.9
75				.242	— .495		
	19	.014	.016			.033	3.0
80				.258	— .462		
	21	.016	.017			.037	2.7
85				.275	— .425		
	19	.014	.018			.037	2.7
90				.293	— .388		
	38	.028	.019			.040	2.5
95				.312	— .348		
	21	.016	.020			.038	2.6
100				.332	— .310		
	25	.019	.020			.040	2 5
105				.352	— .270		
	23	.017	.022			.042	2.4
110				.374	— .228		
	43	.032	.025			.045	2.2
115				.399	— .183		
	34	.025	.029			.053	1.9
120				.428	— .130		

x in mm.	Numb.	Frequ.	Smoothed.	Scheme.	z by tab. 11.	z'.	$\frac{0.1}{z'}$ by tab. 12.
	46	0.034	0.034			0.060	1.7
125				0.462	− 0.070		
	56	.042	.040			.073	1.37
130				.502	+ .003		
	63	.047	.046			.082	1.22
135				.548	.085		
	65	.049	.054			.098	1.02
140				.602	.183		
	94	.070	.058			.109	0.92
145				.660	.292		
	65	.048	.059			.118	0.85
150				.719	.410		
	83	.062	.055			.122	0.82
155				.774	.532		
	55	.041	.049			.123	0.81
160				.823	.655		
	71	0.053	0.043			.128	0.78
165				.866	.783		
	43	.032	.037			.137	0.73
170				.903	.920		
	46	.034	.030			.140	0.71
175				.933	1.060		
	26	.020	.023			.15	0.67
180				.956	1.21		
	24	.018	.017			.15	0.67
185				.973	1.36		
	16	.012	.012			.17	0.59
190				.985	1.53		
	11	.008	.008			.21	0.48
195				.993	1.74		
	2	.002	.004			.20	0.50
200				.997	1.94		
	2	.001	.002			.26	0.38
205				.999	2.20		
	1	.001	.001				
210				1.000	∞		
	0	.000	.000				
215							
	0	.000	.000				
220							
	2	.001	.000				
225							
Total	1338						

Table 9. Example V. Length of wheat ears.
(fig. 10.)

x in mm.	Numb.	Frequ.	Smoothed.	Scheme.	z by tab. 11.	z'.	$\frac{0.1}{z'}$ by tab. 12.
20.5							
	28	0.075	0.075				
30.5				0.075	— 1.018		
	101	.272	.272			0.740	0.14
40.5				.347	— 0.278		
	30	.081	.081			.140	0.71
50.5				.428	— .138		
	15	.040	.046			.092	1.09
60.5				.474	— .046		
	26	.070	.050			.088	1.14
70.5				.524	+ .042		
	24	.065	.074			.134	0.75
80.5				.598	.176		
	37	.099	,106			.202	0.50
90,5				.704	.378		
	58	.156	.150			.367	0.27
100.5				.854	.745		
	35	.094	.100			.445	0.22
110.5				.954	1.190		
	15	,040	.038			.51	0.20
120.5				.992	1.70		
	2	.005	.005			(.24)	(0.42)
130.5				.997	1.94		
	1	.003	.003				
140.5				1.000	∞		
Total	372						

Table 10. Barometerheights at Den Helder (1876.0—1905.0). (fig. 12).

	Winter (Nov.—Feb.)				Summer (May—Aug.)				Proport.
x in mm.	Obs. frequ.	Scheme.	z_w by tab. 11.	z'_w	Obs. frequ.	Scheme.	z_s by tab. 11.	z'_s	$\frac{z'_w}{z'_s}$
720.5	0.0000	0.0000							
722.5	.0001	.0001	— 2.64						
724.5	.0002	.0003	— 2.43	0.21					
726.5	.0004	.0007	— 2.27	0.16					
728.5	.0007	.0014	— 2.115	.155					
730.5	.0014	.0028	— 1.960	.155					
732.5	.0025	.0053	— 1.807	.153					
734.5	.0038	.0091	— 1.670	.137					
736.5	.0052	.0143	— 1.548	.122					
738.5	.0080	.0223	— 1.420	.128	0.0000	0.0000			
740.5	.0107	.0330	— 1.300	.120	.0004	.0004	— 2.38		

742.5	[illegible]	.0478	− 1.178	[illegible]	[illegible]	.0015	− 2.10	[illegible]	[illegible]	
744.5	.0205	.0683	− 1.053	.125	.0024	.0039	− 1.88	.22	.57	
746.5	.0251	.0934	− 0.933	.120	.0061	.0100	− 1.645	.235	.51	
748.5	.0321	.1255	− 0.812	.121	.0106	.0206	− 1.445	.200	.60	0.550
750.5	.0401	.1656	− 0.687	.125	.0194	.0400	− 1.238	.207	.60	
752.5	.0493	.2149	− 0.558	.129	.0353	.0753	− 1.016	.222	.58	
754.5	.0556	.2705	− 0.432	.126	.0583	.1336	− 0.785	.231	.54⁵	
756.5	.0618	.3323	− 0.307	.125	.0850	.2186	− 0.549	.236	.53	
758.5	.0682	.4005	− 0.178	.129	.1103	.3289	− 0.313	.236	.55	
760.5	.0740	.4745	− 0.045	.133	.1321	.4610	− 0.069	.244	.54⁵	.539
762.5	.0767	.5512	+ 0.091	.136	.1401	.6011	+ 0.181	.250	.54⁵	
764.5	.0767	.6279	+ 0.231	.140	.1369	.7380	+ 0.451	.270	.52	
766.5	.0785	.7064	+ 0.384	.153	.1101	.8481	+ 0.727	.276	.55	
768.5	.0734	.7798	+ 0.545	.161	.0774	.9255	+ 1.020	.293	.55	
770.5	.0673	.8471	+ 0.724	.179	.0446	.9705	+ 1.331	.311	.57⁵	
772.5	.0552	.9023	+ 0.916	.192	.0207	.9908	+ 1.667	.336	.57	.551
774.4	.0396	.9419	+ 1.111	.195	.0074	.9982	+ 2.06	.393	.50	
776.5	.0269	.9688	+ 1.318	.207	.0015	.9997	+ 2.43	.37	.56	
778.5	.0173	.9861	+ 1.554	.236	.0003	1.0000				
780.5	.0087	.9948	+ 1.810	.256	.0000					
782.5	.0036	.9984	+ 2.085	.275						
784.5	.0012	.9996	+ 2.38	.295						
786.5	.0004	1.0000								
	.0000									
								Mean	0.544	

Table

z	0	1	2	3	4	5	6	7	8	9
	S.									
— 0.0	0.500	0.494	0.489	0.483	0.477	0.472	0.466	0.461	0.455	0.449
— 0.1	.444	.438	.433	.427	.423	.416	.410	.405	.400	.394
— 0.2	.389	.383	.378	.372	.367	.362	.357	.351	.346	.341
— 0.3	.336	.331	.325	.320	.315	.310	.305	.300	.295	.291
— 0.4	.286	.281	.276	.272	.267	.262	.258	.253	.249	.244
— 0.5	.240	.235	.231	.227	.223	.218	.214	.210	.206	.202
— 0.6	.198	.194	.190	.186	.183	.179	.175	.172	.168	.165
— 0.7	.161	.158	.154	.151	.148	.144	.141	.138	.135	.132
— 0.8	.129	.126	.123	.120	.117	.115	.112	.109	.107	.104
— 0.9	.102	.099	.097	.094	.092	.090	.087	.085	.083	.081
— 1.0	.079	.077	.075	.073	.071	.069	.067	.065	.063	.062
— 1.1	.060	.058	.057	.055	.053	.052	.050	.049	.048	.046
— 1.2	.045	.044	.042	.041	.040	.039	.037	.036	.035	.034
— 1.3	.033	.032	.031	.030	.029	.028	.027	.026	.025	.025
— 1.4	.024	.023	.022	.022	.021	.020	.019	.019	.018	.018
— 1.5	.017	.016	.016	.015	.015	.014	.014	.013	.013	.012
— 1.6	.012	.011	.011	.011	.010	.010	.009	.009	.009	.008
— 1.7	.008	.008	.007	.007	.007	.007	.006	.006	.006	.006
— 1.8	.005	.005	.005	.005	.005	.004	.004	.004	.004	.004
— 1.9	.004	.003	.003	.003	.003	.003	.003	.003	.003	.002
— 2.0	.002	.002	.002	.002	.002	.002	.002	.002	.002	.002
— 2.1	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001
— 2.2	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001
— 2.3	.001	.001	.001	.000	.000	.000	.000	.000	.000	.000

1.

z	0	1	2	3	4	5	6	7	8	9
					S.					
0.0	0.500	0.506	0.511	0.517	0.523	0.528	0.534	0.539	0.545	0.551
0.1	.556	.562	.567	.573	.578	.584	.590	.595	.600	.606
0.2	.611	.617	.622	.628	.633	.638	,643	.649	.654	.659
0.3	.664	.669	.675	.680	.685	.690	.695	.700	.705	.709
0.4	.714	.719	.724	.728	.733	.738	.742	.747	.751	.756
0.5	.760	.765	.769	.773	.777	.782	.786	.790	.794	.798
0.6	.802	.806	.810	.814	.817	.821	.825	.828	.832	.835
0.7	.839	.842	.846	.849	.852	.856	.859	.862	.865	.868
0.8	.871	.874	.877	.880	.883	.885	.888	.891	.893	.896
0.9	.898	.901	.903	.906	.908	.910	.913	.915	.917	.919
1.0	.921	.923	.925	.927	.929	.931	.933	.935	.937	.938
1.1	.940	.942	.943	.945	.947	.948	.950	.951	.952	.954
1.2	.955	.956	.958	.959	.960	.961	.963	.964	.965	.966
1.3	.967	.968	.969	.970	.971	.972	.973	.974	.975	.975
1.4	.976	.977	.978	.978	.979	.980	.981	.981	.982	.982
1.5	.983	.984	.984	.985	.985	.986	.986	.987	.987	.988
1.6	.988	.989	.989	.989	.990	.990	.991	.991	.991	.992
1.7	.992	.992	.993	.993	.993	.993	.994	.994	.994	.994
1.8	.995	.995	.995	.995	.995	.996	.996	.996	.996	.996
1.9	.996	.997	.997	.997	.997	.997	.997	.997	.997	.998
2.0	.998	.998	.998	.998	.998	.998	.998	.998	.998	.998
2.1	.999	.999	.999	.999	.999	.999	.999	.999	.999	.999
2.2	.999	.999	.999	.999	.999	.999	.999	.999	.999	.999
2.3	.999	.999	.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000

For table in 4 decimals see 2nd paper.

$$\frac{0.1}{z'}$$

Tabl

z'	0	1	2	3	4	5	6	7	8	9
0.00	∞	100.0	50.0	33.3	25.0	20.0	16.7	14,3	12.5	11.1
.01	10.00	9.09	8.33	7.69	7.14	6.67	6.25	5.88	5.56	5.26
.02	5.00	4.76	4.55	4.35	4.17	4.00	3.85	3.70	3.57	3.45
.03	3,33	3.23	3.12	3.03	2.94	2.86	2.78	2.70	2.63	2.56
.04	2.50	2.44	2.38	2.33	2.27	2.22	2.17	2.13	2.08	2.04
.05	2.00	1.96	1.92	1.89	1.85	1.82	1.79	1.75	1.72	1.69
.06	1.67	1.64	1.61	1.59	1.56	1.54	1.52	1.49	1.47	1.45
.07	1.43	1.41	1.39	1.37	1.35	1.33	1.32	1.30	1.28	1.27
.08	1.25	1.23	1.22	1.20	1.19	1.18	1.16	1.15	1.14	1.12
.09	1.11	1.10	1.09	1.08	1.06	1.05	1.04	1.03	1.02	1.01
.10	1.000	0.990	0.980	0.971	0.962	0.952	0.943	0.935	0.926	0.91
.11	0.909	.901	.893	.885	.877	.870	.862	.855	.847	.84
.12	.833	.826	.820	.813	.806	.800	.794	.787	.781	.77
.13	.769	.763	.758	.752	.746	.741	.735	.730	.725	.71
.14	.714	.709	.704	.699	.694	.690	.685	.680	.676	.67
.15	0.667	.662	.658	.654	.649	.645	.641	.637	.633	.62
.16	.625	.621	.617	.613	.610	.606	.602	.599	.595	.59
.17	.588	.585	.581	.578	.575	.571	.568	.565	.562	.55
.18	.556	.552	.549	.546	.543	.541	.538	.535	.532	.52
.19	.526	.524	.521	.518	.515	.513	.510	.508	.505	.50
.20	0.500	.498	.495	.493	.490	.488	.485	.483	.481	.47
.21	.476	.474	.472	.469	.467	.465	.463	.461	.459	.45
.22	.455	.452	.450	.448	.446	.444	.442	.441	.439	.43
.23	.435	.433	.431	.429	.427	.426	.424	.422	.420	.41
.24	.417	.415	.413	.412	.410	.408	.407	.405	.403	.40
.25	0.400	.398	.397	.395	.394	.392	.391	.389	.388	,38
.26	.385	.383	.382	.380	.379	.377	.376	.375	.373	.37
.27	.370	.369	.368	.366	.365	.364	.362	.361	.360	.35
.28	.357	.356	.355	.353	.352	.351	.350	.348	.347	.34
.29	.345	.344	.342	.341	.340	.339	.338	.337	.336	.33
.30	0.333	.332	.331	.330	.329	.328	.327	.326	.325	.32
.31	.323	.322	.321	.319	.318	.317	.316	.315	.314	.31
.32	.312	.312	.311	.310	.309	.308	.307	.306	.305	.30
.33	.303	.302	.301	.300	.299	.299	.298	.297	.296	.29
.34	.294	.293	.292	.292	.291	.290	.289	.288	.287	,28
.35	0.286	.285	.284	.283	.282	.282	.281	.280	.279	.27
.36	.278	.277	.276	.275	.275	.274	.273	.272	.272	.27
.37	.270	.270	.269	.268	.267	.267	.266	.265	.265	.26
.38	.263	.262	.262	.261	.260	.260	.259	.258	,258	.25
.39	.256	.256	.255	.254	.254	.253	.253	.252	.251	.25
.40	0.250	.249	.249	.248	.248	.247	.246	.246	.245	.24
.41	.244	.243	.243	.242	.242	.241	.240	.240	.239	.23
.42	.238	.238	.237	.236	.236	.235	.235	.234	.234	.23
.43	.233	.232	.231	.231	.230	.230	.229	.229	.228	.22
.44	.227	.227	.226	.226	.225	.225	.224	.224	.223	.22
.45	0.222	.222	.221	.221	.220	.220	.219	.219	.218	.21
.46	.217	.217	.216	.216	.216	.215	.215	.214	.214	.21
.47	.213	.212	.212	.211	.211	.211	.210	.210	.209	.20
.48	.208	.208	.207	.207	.207	,206	.206	,205	.205	.20
.49	.204	.204	.203	.203	.202	.202	.202	.201	.201	.20

For more extensive tables see:

$$\frac{0.1}{z'}$$

z'	0	1	2	3	4	5	6	7	8	9
50	0.200	0.200	0.199	0.199	0.198	0.198	0.198	0.197	0.197	0.196
51	.196	.196	.195	.195	.195	.194	.194	.193	.193	.193
52	.192	.192	.192	.191	.191	.190	.190	.190	.189	.189
53	.189	.188	.188	.188	.187	.187	.187	.186	.186	.186
54	.185	.185	.185	.184	.184	.183	.183	.183	.182	.182
55	0.182	.181	.181	.181	.181	.180	.180	.180	.179	.179
56	.179	.178	.178	.178	.177	.177	.177	.176	.176	.176
57	.175	.175	.175	.175	.174	.174	.174	.173	.173	.173
58	.172	.172	.172	.172	.171	.171	.171	.170	.170	.170
59	.169	.169	.169	.169	.168	.168	.168	.168	.167	.167
60	0.167	.166	.166	.166	.166	.165	.165	.165	.164	.164
61	.164	.164	.163	.163	.163	.163	.162	.162	.162	.162
62	.161	.161	.161	.161	.160	.160	.160	.159	.159	.159
63	.159	.158	.158	.158	.158	.157	.157	.157	.157	.156
64	.156	.156	.156	.156	.155	.155	.155	.155	.154	.154
65	0.154	.154	.153	.153	.153	.153	.152	.152	.152	.152
66	.152	.151	.151	.151	.151	.150	.150	.150	.150	.149
67	.149	.149	.149	.149	.148	.148	.148	.148	.147	.147
68	.147	.147	.147	.146	.146	.146	.146	.146	.145	.145
69	.145	.145	.145	.144	.144	.144	.144	.143	.143	.143
70	0.143	.143	.142	.142	.142	.142	.142	.141	.141	.141
71	.141	.141	.140	.140	.140	.140	.140	.139	.139	.139
72	.139	.139	.139	.138	.138	.138	.138	.138	.137	.137
73	.137	.137	.137	.136	.136	.136	.136	.136	.136	.135
74	.135	.135	.135	.135	.134	.134	.134	.134	.134	.134
75	0.133	.133	.133	.133	.133	.132	.132	.132	.132	.132
76	.132	.131	.131	.131	.131	.131	.131	.130	.130	.130
77	.130	.130	.130	.129	.129	.129	.129	.129	.129	.128
78	.128	.128	.128	.128	.128	.127	.127	.127	.127	.127
79	.127	.126	.126	.126	.126	.126	.126	.125	.125	.125
80	0.125	.125	.125	.125	.124	124	.124	.124	.124	.124
81	.123	.123	.123	.123	.123	.123	.123	.122	.122	.122
82	.122	.122	.122	.122	.121	.121	.121	.121	.121	.121
83	.120	.120	.120	.120	.120	.120	.120	.119	.119	.119
84	.119	.119	.119	.119	.118	.118	.118	.118	.118	.118
85	0.118	.118	.117	.117	.117	.117	.117	.117	.117	.116
86	.116	.116	.116	.116	.116	.116	.115	.115	.115	.115
87	.115	.115	.115	.115	.114	.114	.114	.114	.114	.114
88	.114	.114	.113	.113	.113	.113	.113	.113	.113	.112
89	.112	.112	.112	.112	.112	.112	.112	.111	.111	.111
90	0.111	.111	.111	.111	.111	.110	.110	.110	.110	.110
91	.110	.110	.110	.110	.109	.109	.109	.109	.109	.109
92	.109	.109	.108	.108	.108	.108	.108	.108	.108	.108
93	.108	.107	.107	.107	.107	.107	.107	.107	.107	.106
94	.106	.106	.106	.106	.106	.106	.106	.106	.105	.105
95	0.105	.105	.105	.105	.105	.105	.105	.104	.104	.104
96	.104	.104	.104	.104	.104	.104	.104	.103	.103	.103
97	.103	.103	.103	.103	.103	.103	.102	.102	.102	.102
98	.102	.102	.102	.102	.102	.102	.101	.101	.101	.101
99	.101	.101	.101	.101	.101	.101	.100	.100	.100	.100

a. Same extent as the present but 4 decimals in L. Zimmermann. Rechentafeln 5 pages) Liebenwerda 1897.

b. Very extensive, 7 decimals. W. H. Oakes. Table of the reciprocals of numbers om 1 to 100 000 (205 pages). London Ch. & E. Layton.

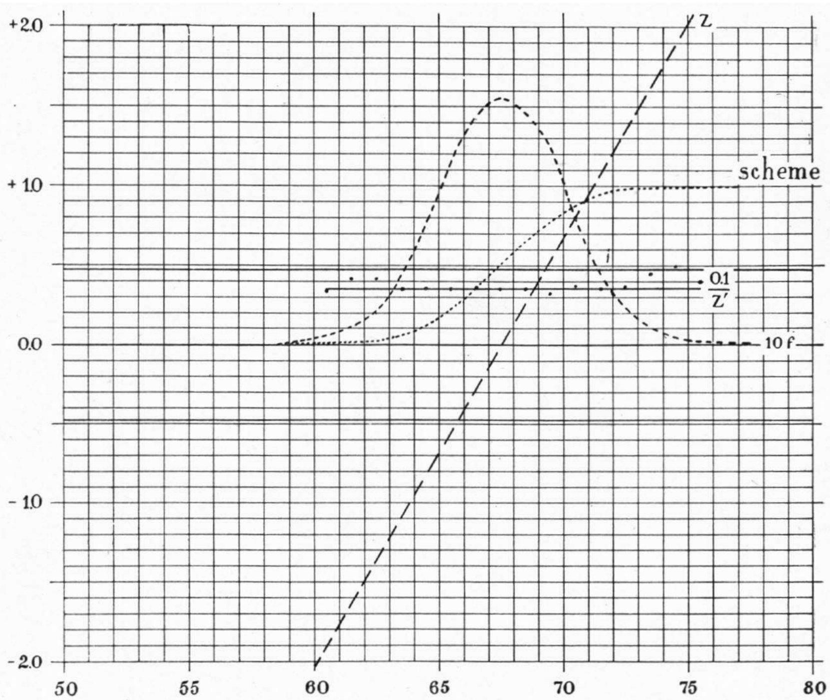

Fig. 1. Stature of 8585 men (tab. 1 and 4).

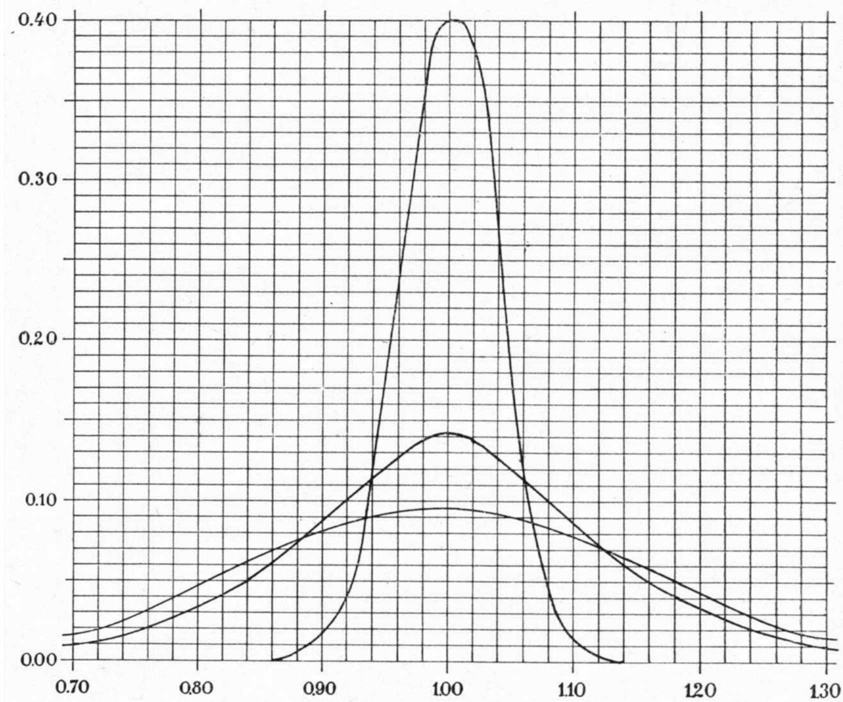

Fig. 2. Observed frequency curves (art. 2).

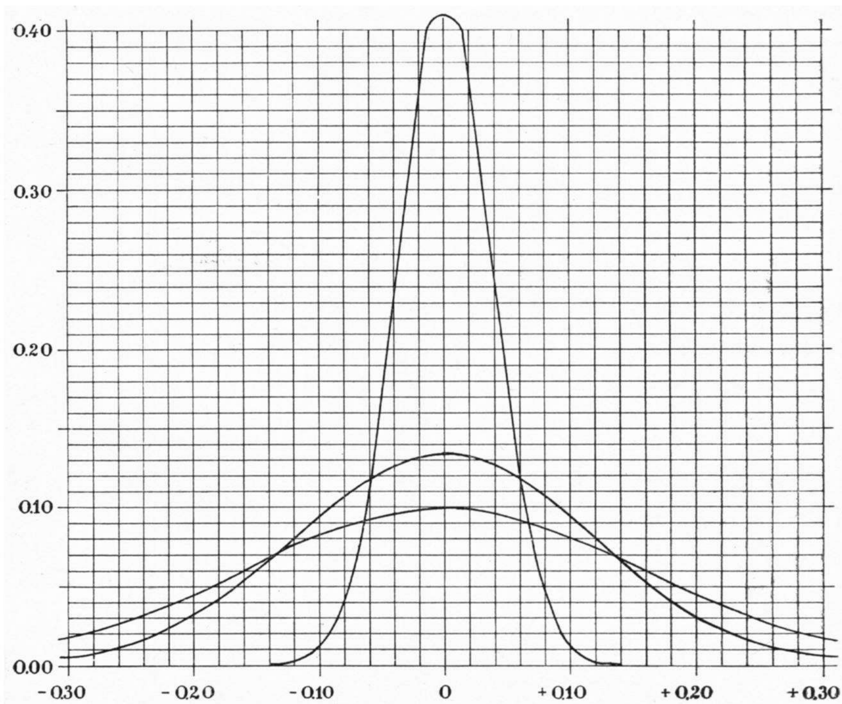

Fig. 3. Theoretical normal curves (art. 2).

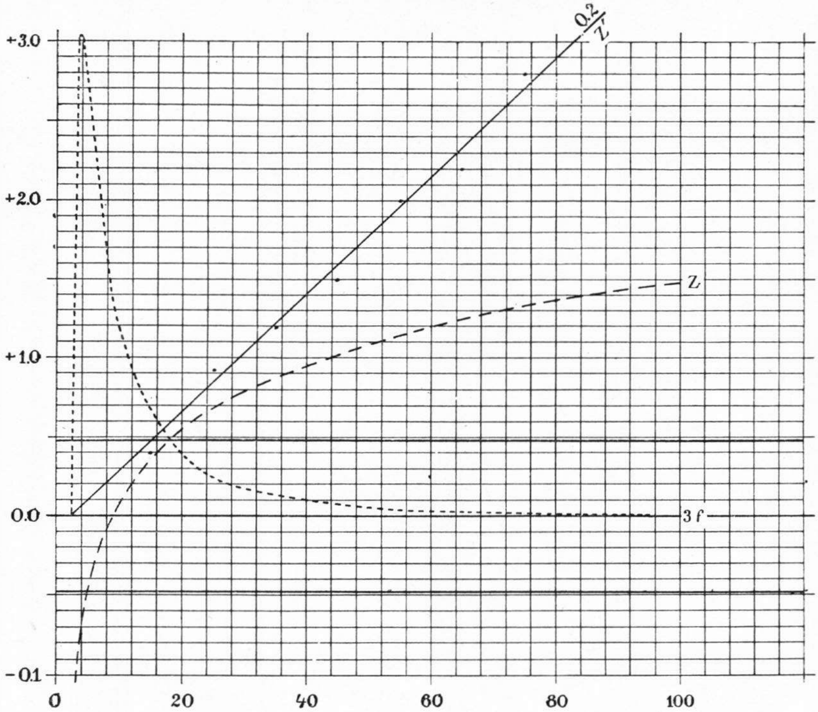

Fig. 4. Valuation of house-property (tab. 6).

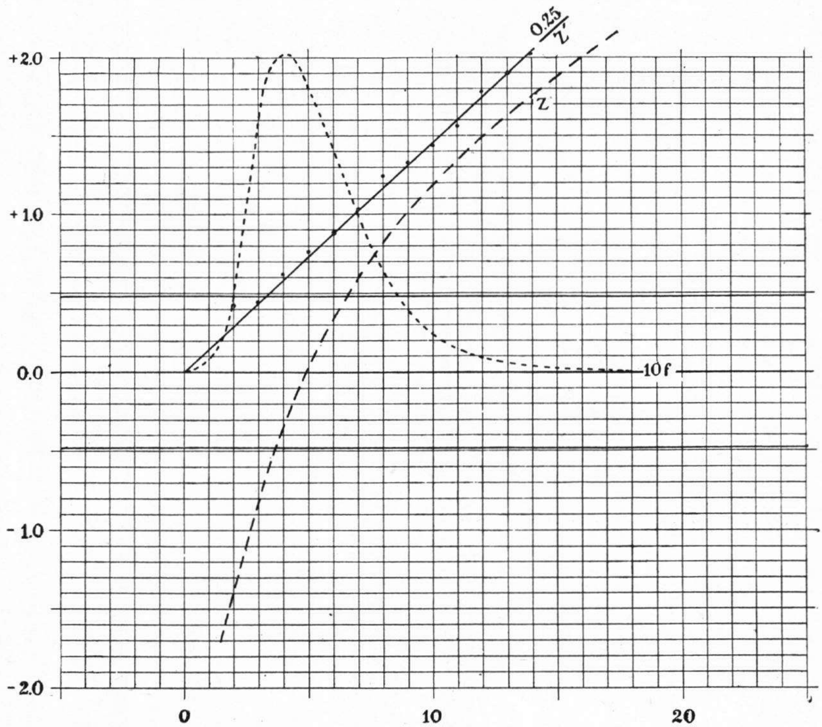

Fig. 5. Threshold of sensation (tab. 5).

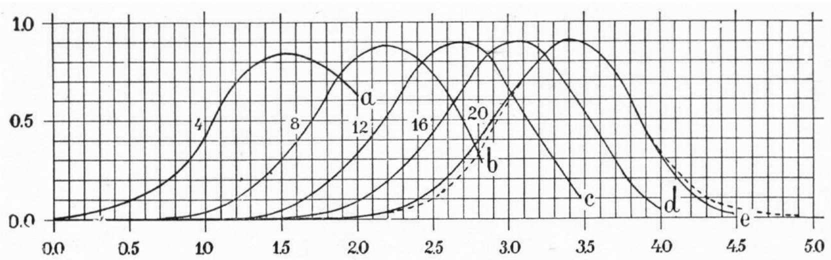

Fig. 6. Normal curve limit to dissymmetrical Point Binomials (art. 7).

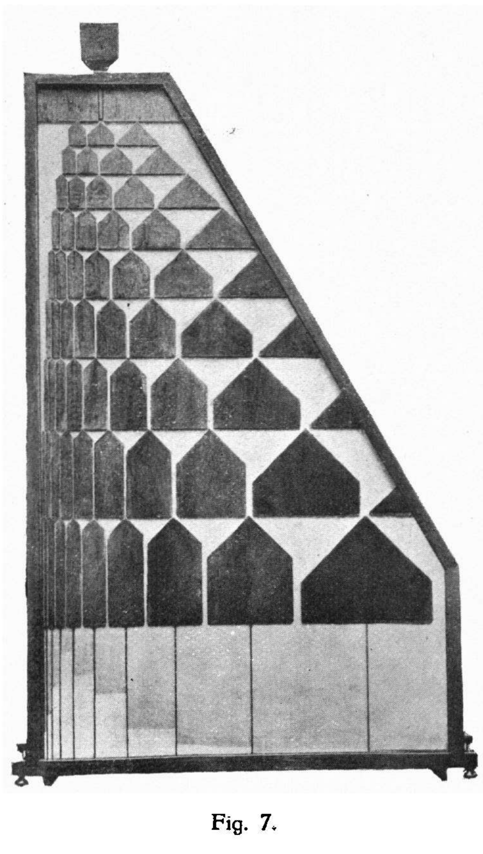

Fig. 7.

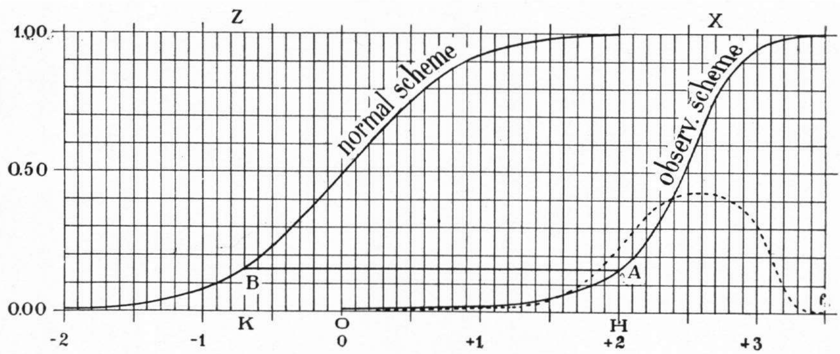

Fig. 8. Questions *a* and *b* art. 13 (tab. 2).

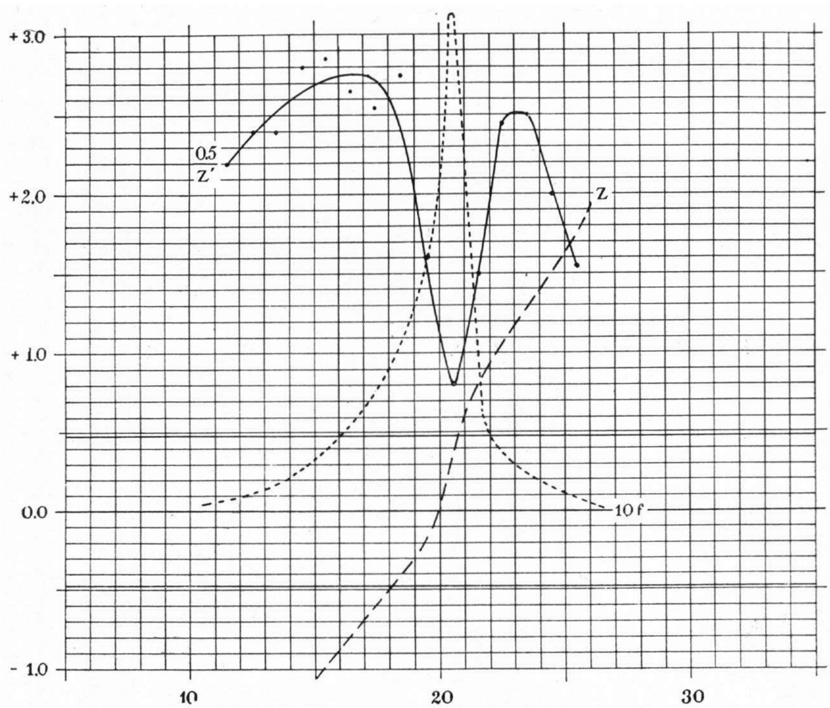

Fig. 9. Diam. of spores of Mucor Mucedo (tab. 7).

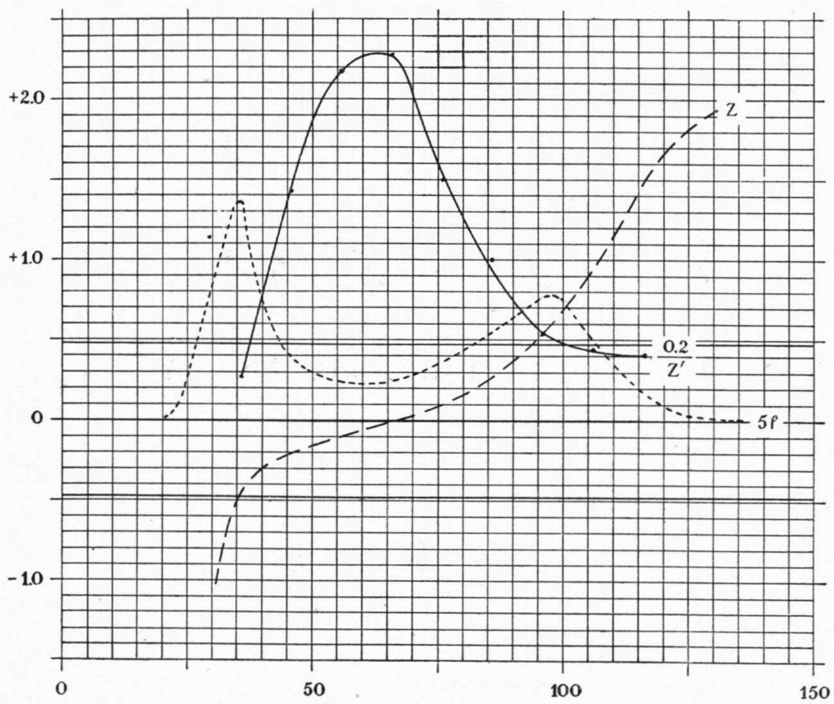

Fig. 10. Length of wheat ears (tab. 9).

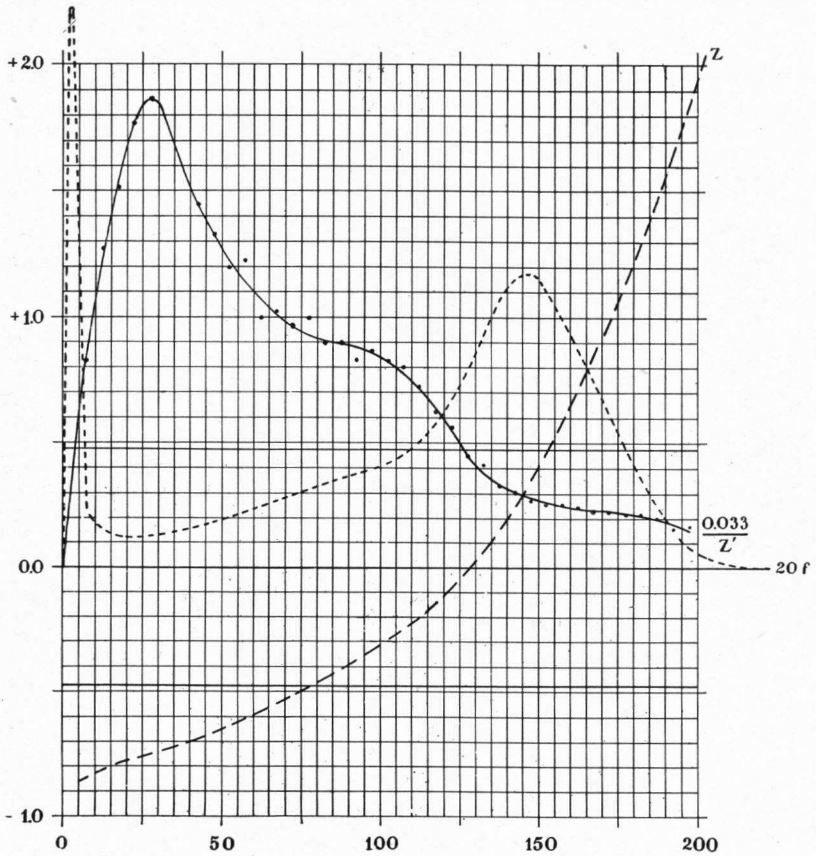

Fig. 11. Stalk-lengths of Linum crepitans (tab. 8).

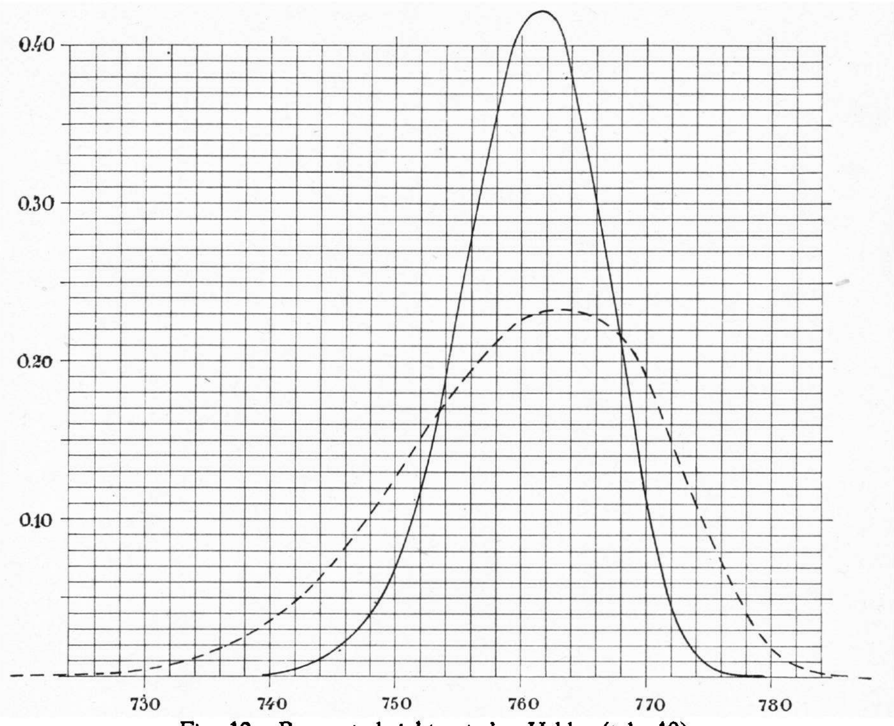

Fig. 12. Barometerheights at den Helder (tab. 10).